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Primordial Subgroups for mod(kG)

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0. INTRODUCTION

Recall from [5] that, given a finite group G and a ring G-functor R, a subgroup H of G is called a primordial subgroup for R if

$$\sum_{L \subsetneq H} t_L^H(R(L)) \subsetneq R(H).$$

Let $\mathscr{P}(R)$ denote the set of primordial subgroups of R and let $\overline{\mathscr{P}(R)}$ denote the subgroup closure of $\mathscr{P}(R)$. In [5, 2.1], it is proved that $\overline{\mathscr{P}(R)}$ is the minimal set $\mathscr{D}$ of subgroups of G, closed under conjugation and formation of subgroups, satisfying the condition

$$\sum_{H \in \mathscr{D}} t_H^G(R(H)) = R(G).$$

Let k be any field and consider the ring G-functor R_k, which assigns to each $H \subseteq G$ the Grothendieck ring of the category of kH-modules with the usual restriction, induction (denoted by t) and conjugation. It is well known that for fields k of zero characteristic, $\mathscr{P}(R_k)$ equals the set of k-elementary subgroups of G; Thévenaz conjectured in [5] that this is also the case for fields of positive characteristic (see definitions below). In this work we give a short proof of this statement. First we show for the ring G-functor ${}_{\mathbb{Q}}R_k = \mathbb{Q} \otimes_{\mathbb{Z}} R_k$ that $\mathscr{P}({}_{\mathbb{Q}}R_k)$ equals the set of cyclic subgroups of G of order prime to the characteristic of k. Then we use this to prove Thévenaz conjecture.

1. NOTATION AND PRELIMINARY LEMMAS

Fix a field k of characteristic $p \geqslant 0$. Let n be a positive integer (prime to p if $p > 0$) and let ω_n be an nth-primitive root of unity in some algebraic closure of k. Let $k_n = k[\omega_n]$. Then k_n/k is a Galois extension of k. Let $\mathscr{G}_n =$

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$\mathrm{Gal}(k_n/k)$. Then there is a monomorphism $\mathscr{G}_n \to (\mathbb{Z}/n\mathbb{Z})^*$. We consider $\mathscr{G}_n$ as a subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$.

We say that a finite group H is *k-elementary* if $H = \langle x \rangle \rtimes Q$, where Q is a q-group (q a prime number), the order m of x is relatively prime to q (and to p if $p > 0$), and for each $a \in Q$ there is $t \in \mathscr{G}_m$ such that $a^{-1}xa = x^t$ (c.f. [1, Sect. 21A]).

Note that in the last definition we can replace $\mathscr{G}_m$ by $\mathscr{G}_n$, where n is any multiple of m (prime to p if $p > 0$). From now on, G denotes a fixed finite group, $p > 0$, n denotes the p'-part of $|G|$, and k_0 is the prime field of k.

The next lemma tells us that we can restrict ourselves to finite fields and therefore to p-modular systems in the sense of [1, Sect. 16A].

LEMMA 1. *Let $k' = k_0[\omega_n]$ and $E = k \cap k'$. Then $R_E \cong R_k$ as ring G-functors (the isomorphism sends the class of the EG-module M to the class of $k \otimes_E M$).*

Proof. Let $H \subseteq G$ and let $M_1, ..., M_s$ be a set of representatives of the isomorphism classes of the simple $k'H$-modules. Then $k_n \otimes_{k'} M_1, ..., k_n \otimes_{k'} M_s$ is a set of representatives of the isomorphism classes of the simple k_nH-modules. Clearly $\mathscr{G}_n \cong \mathrm{Gal}(k'/E)$. The group $\mathrm{Gal}(k'/E)$ acts on the simple $k'H$-modules and the orbits correspond to the simple EH-modules; in fact, if $\mathcal{O}$ is an orbit then $N_{\mathcal{O}} := \bigoplus_{M \in \mathcal{O}} M$ is a simple EG-module (see [2, 74.9]). The kH-simple modules are obtained similarly from the k_nH-modules. Therefore if $N_1, ..., N_t$ is a set of representatives of the isomorphism classes of the EH-modules then $k \otimes_E N_1, ..., k \otimes_E N_t$ is a set of representatives of the isomorphism classes of the kH-modules, so the lemma is proved. ∎

In order to calculate $\mathscr{P}(R_k)$ we may assume that $k \subseteq k_0[\omega_n]$. Then there is a p-modular system (K, R, k), with R a complete discrete valuation ring and K (a finite extension of the p-adic field) the quotient ring of R. There is another p-modular system (K_n, R_n, k_n), where $K_n = K[\omega'_n]$ and $R_n = R[\omega'_n]$, with ω'_n an nth-root of unity mapping onto ω_n. It is known that $\mathrm{Gal}(K_n/K) \cong \mathscr{G}_n$.

LEMMA 2. *The primordial set $\mathscr{P}({}_{\mathbb{Q}}R_k)$ equals the set $\mathscr{C}$ of cyclic subgroups of G of order prime to p.*

Proof. First assume $k = k_n$. From the Dress Induction Theorem (see [6]) and the isomorphism of ring G-functors $R_k \cong \mathrm{Brch}(G)$ (see [1, 17.14]), where $\mathrm{Brch}(G)$ denotes the ring of Brauer characters of G, we deduce that

$$ {}_{\mathbb{Q}}R_k(G) = \sum_{H \in \mathscr{C}} t_H^G({}_{\mathbb{Q}}R_k(K)). $$

Therefore, by minimality, we have $\mathscr{P}({}_{\mathbb{Q}}R_k) \subseteq \mathscr{C}$.

Now, for $H \in \mathscr{C}$, since p does not divide $|H|$, we have that R_k is isomorphic as a ring H-functor to the ordinary character ring and it is known that H itself is primordial for ${}_{\mathbb{Q}}R_k$ (see [4, 3.13]). Therefore $H \in \mathscr{P}({}_{\mathbb{Q}}R_k)$.

In general there is a monomorphism of G-functors ${}_{\mathbb{Q}}R_k \hookrightarrow {}_{\mathbb{Q}}R_{k_n}$. Therefore by [5, 5.2], $\mathscr{P}({}_{\mathbb{Q}}R_k) = \mathscr{P}({}_{\mathbb{Q}}R_{k_n}) = \mathscr{C}$. ∎

2. Proof of the Theorem: $\mathscr{P}(R_k) = \mathscr{E}_k$.

Consider the decomposition map of ring G-functors d: $R_K \to R_k$ (see [1, Sect. 16]). Then by [5, 2.5], $\mathscr{P}(R_k) \subseteq \mathscr{P}(R_K)$; by [1, 21.6] and [3], the latter equals $\mathscr{E}_K$. On the other hand for all $H \subseteq G$ it is easy to prove that the ring ${}_{\mathbb{Q}}R_k(H)$ has no nilpotent elements (see [1, Sect. 17, Exercise 4]) and so the inclusion of ring G-functors $R_k \hookrightarrow {}_{\mathbb{Q}} R_k$ together with Lemma 2 give $\mathscr{P}(R_k) \subseteq \mathscr{H}(\mathscr{C})$, where $\mathscr{H}(\mathscr{C})$ is the set of hyperelementary subgroups of G whose normal cyclic subgroup is in $\mathscr{C}$ (see [5, 5.1]).

We claim that $\mathscr{H}(\mathscr{C}) \cap \mathscr{E}_K = \mathscr{E}_k$. The inclusion $\mathscr{H}(\mathscr{C}) \cap \mathscr{E}_K \supseteq \mathscr{E}_k$ is trivial. Suppose $H \in \mathscr{H}(\mathscr{C}) \cap \mathscr{E}_K$. Then, since $H \in \mathscr{H}(\mathscr{C})$, we have $H = \langle x \rangle \rtimes Q$, where Q is a q-group and p and q do not divide the order of x. On the other hand, since $H \in \mathscr{E}_k$, then $H = \langle y \rangle \rtimes D$, where D is a p_0 group and p_0 (a prime number) does not divide the order of y. We have three cases:

(a) $p_0 = p$. In this case p does not divide the order of y and so $H \in \mathscr{E}_k$ by the remarks above.

(b) $p_0 = q$. In this case the order of x equals the order of y, so p does not divide the order of y and so, as in (a), $H \in \mathscr{E}_k$.

(c) $p_0 \neq p$ and $p_0 \neq q$. In this case there exists $g \in H$ such that ${}^gD \subseteq \langle x \rangle$ and, since clearly ${}^gD$ is a characteristic subgroup of $\langle x \rangle$ which is normal in H, we conclude that D itself is normal in H, so H is cyclic and therefore $H \in \mathscr{E}_k$. This proves the claim.

Now let $H \in \mathscr{E}_k$, say, $H = \langle x \rangle \rtimes Q$ with Q a q-group. If $q \neq p$ then the decomposition map $d: R_K \to R_k$ is an isomorphism of ring H-functors. Therefore we have an isomorphism

$$\frac{R_K(H)}{\sum_{L \subsetneq H} t_L^H(R_K(L))} \cong \frac{R_k(H)}{\sum_{L \subsetneq H} t_L^H(R_K(L))}.$$

Since it is known that the left-hand side is different from zero, we have $H \in \mathscr{E}_k$.

Now assume $q = p$ and let $L \subsetneq H$. Let us prove that for each $\varphi \in R_k(L)$ we have $t_L^H(\varphi)(x) \in p\mathbb{Z}[\omega_n]$. (Here we consider the elements of $R_k(L)$ as

Brauer characters.) From the transitivity of induction we can assume without loss of generality that either $L = \langle x \rangle \rtimes Q_1$ with $Q_1 \subsetneq Q$ or else $L = \langle x^r \rangle \rtimes Q$ with $\langle x^r \rangle \subsetneq \langle x \rangle$. In the first case

$$t_L^H(\varphi)(x) = \sum_{a \in Q/Q_1} \varphi(a^{-1}xa) = \varphi(x)[Q : Q_1] \in p\mathbb{Z}[\omega_n];$$

the last equality holds since φ is constant on k-conjugacy classes (see [1; step 2 in the proof of 21.25]). In the second case $t_L^H(\varphi)(x)$ is clearly zero. We conclude that $H \in \mathscr{P}(R_k)$.

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